Winding numbers and Fourier series

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This is an expository talk on a topic of classical analysis, arizing from the BMO–theory of topological degree (Brézis–Nirenberg 1995) [6]. We sketch the history of the subject and some of its recent developments.

1 The starting point

In October 1995 Haïm Brézis visited Rutgers University and gave a lecture at the seminar of Israel Gelfand on his recent work with Louis Nirenberg on the extension of the notion of topological degree to a class of functions larger than the class of continuous functions, namely VMO, the class of functions with vanishing mean oscillation. Vanishing mean oscillation is expressed by the formula

$$\lim_{|B|\to 0} \frac{1}{|B|^2} \iint_{B\times B} |f(x)-f(y)|\, dx\, dy = 0$$

when f maps an open set $\Omega \subset \mathbb{R}^m$ into $\mathbb{R}^n$, B denotes a ball contained in Ω and $|B|$ its volume, $dx = dx_1 \cdots dx_m$ and $|z| = |z_1^2 + \cdots + z_n^2|^{1/2}$. When f maps a manifold into a manifold, this definition should be translated accordingly.

Gelfand was eager to have examples, and Brézis had plenty of them. First,

$$W^{s,p}(\Omega) \subset VMO(\Omega)$$

when $0 < s < 1$, $1 < p < \infty$, $sp = n$, and $f \in W^{s,p}(\Omega)$ means

$$\iint_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dxdy < \infty\,.$$

Then, the Sobolev classes

$$H^{n/2}(\Omega) \subset VMO(\Omega)$$

since $H^s = W^{s,2}$. In particular, going to mapping of the n–sphere into itself,

$$H^1(S^2, S^2) \subset VMO(S^2)\,,$$

the original motivation of Brézis [5]. In the same way,

$$H^{1/2}(S^1, S^1) \subset VMO(S^1)\,.$$

Gelfand felt satisfied with S^1, but not with the definition of $H^{1/2}$ by means of a double integral. Since f can be expressed as $f(e^{it})$, $t \in \mathbb{R}$, and $|f(e^{it})| = 1$, how to write the definition in terms of the Fourier coefficients of f,

$$a_n = \int f(e^{it})e^{-int}\frac{dt}{2\pi}$$

(here and in the sequel we write $\int$ instead of $\int_0^{2\pi}$). Brézis had the answer immediately : the definition can be expressed as

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 < \infty\,.$$

Then Gelfand asked another question. Can you express the topological degree by means of the a_n ? Brézis could not answer on the spot. He went home, made a little computation, and got the simple and beautiful formula

$$\deg f = \sum_{-\infty}^{\infty} n|a_n|^2\,.$$

Here $\deg f$ denotes the usual topological degree if f is continuous, and the VMO–degree in general, that is, $f \in H^{1/2}$.

This was integrated in the article of Brézis and Nirenberg [6] and initiated a series of other questions. The first is already in [6]. Most of them can be found in the "mise au point" by Brézis in 2006 [3]. We shall see some answers and new questions.

Question 1 [6].

What happens when f is continuous (then $\deg f$ exists) and does not belong to $H^{1/2}(S^1, S^1)$, that is

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 = \infty\ ?$$

Is there any summation process for the series

$$\sum_{-\infty}^{\infty} n\,|a_n|^2$$

such that $\deg f$ can be computed in that way ?

The first answers were given by Jacob Korevaar in 1999 [10]. Korevaar considers two summation processes, namely

$$\lim_{n\to\infty} \sum_{-n}^{n} m|a_m|^2\,,$$

using symmetrical partial sums, and

$$\lim_{r\nearrow 1} \sum_{-\infty}^{\infty} r^{|m|}|a_m|^2\,,$$

the process of Abel–Poisson. The second is stronger than the first. Korevaar shows that they work when f has bounded variation, $f \in C \cap BV$, but that none of them work under the mere assumption $f \in C$. Actually they diverge for some f, and they converge to a value different from $\deg f$ for some other f.

That led to another question.

Question 2 [3].

Does $\deg f$ depend on the absolute value of the Fourier coefficients a_n only ? Since the energy of f is defined by the $|a_n|^2$, and the topological degree of a mapping from S^1 to S^1 is nothing but the winding number, the question can be asked in a pleasant form : can we hear the winding number ?

The answer is negative. Jean Bourgain and Gaby Kozma were able to construct two functions belonging to $C(S^1, S^1)$ with the same $|a_n|$ and different degrees [2]. It is a very difficult construction.

Question 3 [8, 9].

Let us return to the summation processes. Let us begin by the Abel–Poisson process, since it is stronger than most usual processes. If we assume that f satisfies a Hölder condition of order $\alpha > 0$, that is, in Zygmund's notation, $f \in \Lambda_\alpha(S^1, S^1)$ (Zygmund, as many authors, says "Lipschitz condition of order α") [12], is it true that

$$\deg f = \lim_{r\uparrow 1} \sum_{-\infty}^{\infty} n\; r^{|n|}|a_n|^2 \text{ ?}$$

The answer is positive for $\alpha > \frac{1}{3}$ and negative for $\alpha \leq \frac{1}{3}$. This comes from a more precise statement, which involves the classes λ_α^p of Zygmund ([12], p. 45), defined as

$$\lambda_\alpha^p = \{g : \int |g(t+s) - g(s)|^p ds = o(t^\alpha),\ t \downarrow 0\}\,,$$

and a non–classical summation process, namely

$$\lim_{t\to 0} \sum_{-\infty}^{\infty} |a_n|^2 \frac{\sin nt}{t}\,.$$

This limit exists and is equal to $\deg f$ when $f \in C \cap \lambda_{1/3}^3(S^1, S^1)$ but there exists $f \in \Lambda_{1/3}(S_1, S_1)$ such that it doesn't exist, or it exists and is different from $\deg f$ [8].

The positive part of this statement is valid when $C \cap \lambda_{1/3}^3(S^1, S^1)$ is replaced by $W_{1/3}^3(S^1, S^1)$ [3]. We don't know if it is still valid under the assumption $f \in VMO \cap \lambda_{1/3}^3(S^1, S^1)$; that would provide a commun generalization to both statements.

Question 4 [9].

We introduced three summation processes, and there are many others. It was a popular subject in the 1920's, and the best reference is Hardy's book of 1949, Divergent Series [7]. I returned to this topic in [9].

Hardy considers series of terms indexed by positive integers, say

$$\sum_1^\infty u_m\,.$$

In our situation

$$u_m = m(|a_m|^2 - |a_{-m}|^2)\,.$$

Ordinary convergence to s means

$$(C) \qquad s = \lim_{n\to\infty} \sum_1^n u_m$$

Cesàro summability of order k $(k > -1)$ to s means

$$(C,k) \qquad s = \lim_{n\to\infty} \binom{n+k}{k}^{-1} \sum_m \binom{n+k-m}{k} u_m\,;$$

(C,O) is the same as C, and $(C,1)$ deals with the arithmetic means of partial sums ; it is the process used by Fejér in Fourier series. The processes

$$(R,k) \qquad s = \lim_{r\downarrow 0} \sum_{1}^{\infty} u_m \Big(\frac{\sin mt}{mt}\Big)^k$$

(k being a positive integer) are called Riemann summation processes of order k. The original Riemann process is $(R,2)$ and it was used in order to study everywhere convergent trigonometric series. The process we just used is $(R,1)$. Whenever we consider (R,k) we assume $\sum\limits_{1}^{\infty} |u_m| m^{-k} < \infty$. The process

$$(A) \qquad s = \lim_{n\uparrow 1} \sum_{1}^{\infty} r^m u_m$$

is the Abel, or Abel–Poisson, summation process.

It is classical, and easy to see, that (C,k') is stronger than (C,k) and (R,k') is stronger than (R,k) if $k'>k$, and that (A) is stronger than all (C,k) and stronger than $(R,2)$. We write

$$\begin{array}{lcl} k'>k \implies (C,k) & \longrightarrow & (C,k') \\ k'>k \implies (R,k) & \longrightarrow & (R,k') \end{array}$$

$$\begin{array}{lcl} (C,k) & \longrightarrow & (A) \\ (R,2) & \longrightarrow & (A) \end{array}$$

Hardy mentioned some more subtle results :

$$\begin{array}{lcll} (R,2) & \longrightarrow & (C,2+\delta) & (\delta>0) \\ (R,1) & \longrightarrow & (C,1+\delta) & (\delta>0) \end{array}$$

The first is due to Kuttner (1935) [11] and the second to Zygmund (1928) [13]. Moreover, Kuttner gave an example showing that

$$(R,3) \nrightarrow (A)\,.$$

It is easy to see that

$$(R,1) \nrightarrow (C)\,.$$

More interesting is the fact that

$$(R,1) \nrightarrow (C,1)\,.$$

In the opposite direction,

$$(C) \nrightarrow (R,1)\,.$$

These last results can be found at the end of [9].

Question 5 [4, 1].

In September 2008, Brézis made a report on topological degree and asked the following question. Let $f \in C(S^1, S^1)$, with Fourier coefficients $a_n (n \in \mathbb{Z})$. Is it true that

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 \leq |\deg f| + 2\sum_{1}^{\infty} n|a_n|^2\,?$$

Since it is true when the first member is finite, the question can be written as

$$\sum_{1}^{\infty} n\,|a_n|^2 < \infty \Longrightarrow \sum_{-\infty}^{\infty} |n|\,|a_n|^2 < \infty\,?$$

I could answer in a particular case ($f \in \Lambda_\alpha$, $\alpha > 0$) and Bourgain in the general case.

The answer is positive. Moreover

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 \leq 32\sum_{1}^{\infty} n\,|a_n|^2\,.$$

The implication is valid in a more general situation. Let $s > 0$. Then

$$\sum_{1}^{\infty} n^{2s}\,|a_n|^2 < \infty \Longrightarrow \sum_{-\infty}^{\infty} |n|^{2s}\,|a_n|^2 < \infty\,.$$

But, except when $s = \dfrac{1}{2}$, there is no constant $C = C(s)$ such that

$$\sum_{-\infty}^{\infty} |n|^{2s}\,|a_n|^2 < C\sum_{1}^{\infty} n^{2s}\,|a_n|^2 \qquad [1]$$

These results are valid when f is supposed to be VMO instead of continuous, but not when f is supposed to be bounded (counterexample : a Blaschke product).

The study of Fourier series of continuous or VMO unimodular functions is an interesting byproduct of the study of winding numbers.

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